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A MODEL OF A DRIFTING ICE COVER†

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Some theoretical ideas are developed concerning the dynamics of a drifting ice cover modelled by a four-phase two-dimensional continuous medium with elastic–plastic rheology under bulk and shearing deformations, including processes of hummocking and the formation of fractures [1]. Phase transition conditions are formulated in closed form, depending in general on eight functions that characterize the rheological properties of the sheet. New exact solutions are developed for the model equations, describing a diverging ice cover drifting in the field of an ocean eddy, the interaction of a diverging ice cover with surface waves, the plastic flow of a close pack ice cover under the influence of a localized sea current, and the collision of a close pack ice massif with a rigid, wedge-shaped wall.

The choice of a model for ice covers depends on the characteristic space–time scales of the phenomena being studied and on the physical and mechanical properties of ice covers. If the floes are uniformly distributed over the liquid surface and the relative speeds of neighbouring floes are small, the movement and deformations of an ice cover may be described in terms of a continuous medium with complex rheology. [1–7]. Previously proposed models include the elastic–plastic model [2, 3], the viscoplastic model [4], the cavitating fluid model [5], and models with viscous rheology (see, for example, [6, 7]).

The main aim of most studies of ice-cover dynamics, whatever models they adopt, has been the numerical simulation of ice drift in a fixed region, rather than a theoretical investigation of the exact solutions of the model equations. Hence questions relating to the physical meaning of the solutions have also been ignored. Among the few exceptions are the papers [3, 6], which point out the importance of theoretical research.

Pritchard [3], by theoretically analysing the equations of ice-cover dynamics, reached certain conclusions about the relationship between structures in drifting ice fields and the characteristic curves (gliding strips) of the model equations. Gol'dshtein and Mosolov [6] investigated the dependence of ice strength on the fractal characteristics of ice covers, in the context of a ship moving in broken ice.

Our main aim here will be to construct a closed model of an ice cover, describing ice drift, taking into account the processes of hummocking and fracture formation, and to carry out a theoretical investigation of the simplest problems of ice-cover dynamics that have a bearing on actual situations.

1. The fundamental equations of the ice-cover model are [1]

$$d(\rho Ah)/dt + \rho Ah\nabla u = 0 \qquad (1.1)$$

$$\rho Ah(du/dt + f \times u) = AF + \nabla\sigma \qquad (1.2)$$

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$$\frac{ds}{dt} - \tau\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \lambda s = \mu\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$
$$\frac{d\tau}{dt} + s\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \lambda\tau = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (1.3)$$

$$p = p(\rho, A, h) \qquad (1.4)$$
$$\mathbf{u} = (u, v, 0),\ \mathbf{f} = (0, 0, f),\ s = {}^1/_2(\sigma_{xx} - \sigma_{yy}),\ \tau = \sigma_{xy}$$
$$p = -{}^1/_2(\sigma_{xx} + \sigma_{yy}),\ d/dt = \partial/\partial t + \mathbf{u}\nabla,\ \nabla = (\partial/\partial x, \partial/\partial y, 0)$$

where x, y and t are the horizontal coordinates and time, ρ and h are the density and thickness of the ice cover, averaged over an element of area, A and $\mathbf{u}$ are the compactness and drift velocity, σ is the internal stress tensor of the cover, $\mathbf{F}$ are the external forces, f is the Coriolis parameter and $\mu = \mu(A, h)$ is the shear modulus.

Equations (1.1) and (1.2) are the laws of conservation of mass and momentum. The Prandtl–Reuss equations (1.3) determine the rheology of the ice cover under shear deformations [8]. The equation of state (1.4) describes the behaviour of the ice cover under compression and stretching.

The system of six equations (1.1)–(1.4) involves eight unknowns ρ, A, h, u, v, s, τ and p, describing the macroscopic state of an ice-cover element, and a parameter λ, which in plastic shearing deformations is eliminated from (1.3) by using the plasticity condition; in other cases it is simply set equal to zero. Equations (1.1)–(1.4) may be closed in different ways, depending on which of the four phase states of an ice cover is to be represented: diverging, close pack, hummocky or non-hummocky.

A hummocky or non-hummocky ice cover may be in either of the close pack or diverging states. Since a hummocky ice cover consists of pieces of ice floes piled up on one another, an important parameter is the average thickness of the pieces in a hummock: $h_f \leqslant h$. The parameter $h - h_f$ is a measure of the roughness of a hummocky ice cover. The limiting case $h = h_f$ represents an ice cover in which the contacts among the ice floes are such as to enable them to glide over one another without much breakage. Examples of hummocky ice with $h = h_f$ are shown in Figs 1 and 2. Compression crushes the edges of the floes in Fig. 1 in such a way that they may easily glide over one another without further breakage. The gaps between the floes are filled with ice rubble, which acts as a lubricant. The floes in Fig. 2 are plate-shaped, corresponding to young sea ice.

For a diverging ice cover, Eqs (1.1)–(1.4) are closed by adding the relations

$$s = \tau = p = \mu = 0,\ dh/dt = 0,\ 0 < A < A_*$$

The condition for a diverging ice cover to become close packed is

$$A = A_* \leqslant 1$$

A close pack ice cover may experience elastic and plastic deformations. Plastic deformations are of three types: shear, compaction and hummocking. The properties of shear and compaction were described in [1]. We might mention here that when an ice cover experiences compaction its compactness increases without any change in thickness. Compaction is possible when $0 < A < A_{**}$ for a hummocky sheet one puts $A_{**} = 1$.

Breakage involves changes in the microstructure of the ice cover and breakup of the connections between floes. When a close pack non-hummocky ice cover with $A = A_{**}$ is compressed, the floe edges are crushed in such a way that the cover goes into a hummocky state with $h = h_f$. Stretching causes the ice to break and go into a diverging state.

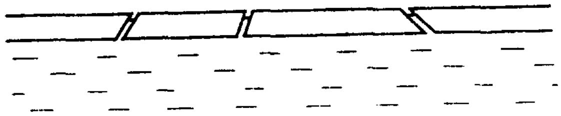

Fig. 1.

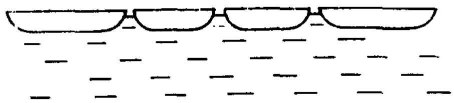

Fig. 2.

The thickness of an ice cover may increase only in a hummocky sheet with $A = A_{**} = 1$ in further hummocking.

The compaction condition for a non-hummocky ice cover and compaction and hummocking conditions for a cover sheet are similar in form

$$p = \pi_p(A, h) > 0,\ dA/dt > 0,\ dh/dt = 0 \qquad (1.5)$$
$$\rho = \rho_{cr}(A, h),\ A_* \leqslant A < A_{**}$$

$$p = \pi_p(A, h, h_f) > 0,\ dA/dt > 0,\ dh/dt = 0 \qquad (1.6)$$
$$\rho = \rho_{cr}(A, h, h_f),\ A_* \leqslant A < 1;$$
$$p = \pi_p(h, h_f) > 0,\ dh/dt > 0,\ \rho = \rho_{cr}(h, h_f),\ A = 1$$

For brevity, we shall henceforth refer to a non-hummocky ice cover with $A \in [A_*,\ A_{**})$ or a hummocky ice cover as an ice cover of type 1, and to a non-hummocky ice cover with $A = A_{**} = 1$, as an ice cover of type 2.

The conditions for plastic shearing and breakage may be expressed as a function [9]

$$|\tau_n| = F(\sigma_n, A, h) \qquad (1.7)$$

where τ_n and σ_n are the shearing and normal stresses in an area element with normal **n**. In a hummocky ice cover F also depends on h_f.

Typical plots of τ_n and σ_n corresponding to (1.7), for fixed A and h, are shown in Figs 3 and 4. The curve $ABCDE$ in Fig. 3 corresponds to plasticity conditions for an ice cover of type 1. The closed curve $ABCDEF$ in Fig. 4 corresponds to plasticity conditions for an ice cover of type 2.

The stresses in an area element with normal **n** may be expressed as follows [10]:

$$\sigma_n + p = R\cos(2\varphi),\ \tau_n = R\sin(2\varphi),\ R = {}^1\!/_2(\sigma_{\max} - \sigma_{\min}) \qquad (1.8)$$

where φ is the angle between **n** and the direction of the principal stress $\sigma_{\max}$. The principal values of the stress tensor σ are defined by the formulae

$$\sigma_{\max} = -p + (\tau^2 + s^2)^{1/2},\ \sigma_{\min} = -p - (\tau^2 + s^2)^{1/2} \qquad (1.9)$$

It follows from (1.8) that the stressed state at a point is characterized by the Mohr circle in the $(\tau_n,\ \sigma_n)$ plane, while condition (1.7) means that the Mohr circle is tangent to the curve $ABCDE$ in Fig. 3 or the curve $ABCDEF$ in Fig. 4.

It is assumed that when the Mohr circle touches the arcs AB and CD, plastic shearing may occur. When the Mohr circle touches the arc BCD, the connections between the ice floes (if such existed) are broken and, under stretching, the cover begins to diverge. When the Mohr circle touches the arc AFE, the transition is into the hummocky state. The Mohr circles O_1 in Figs 3 and 4 correspond to limiting states of plastic shearing, before breakage. The point O_1 has coordinates $(-\pi_1(A, h), 0)$, $\pi_1 < 0$. For a hummocky ice cover the function π_1 also depends on h_f. The Mohr circles O_p correspond to limiting states of plastic shear before compaction or

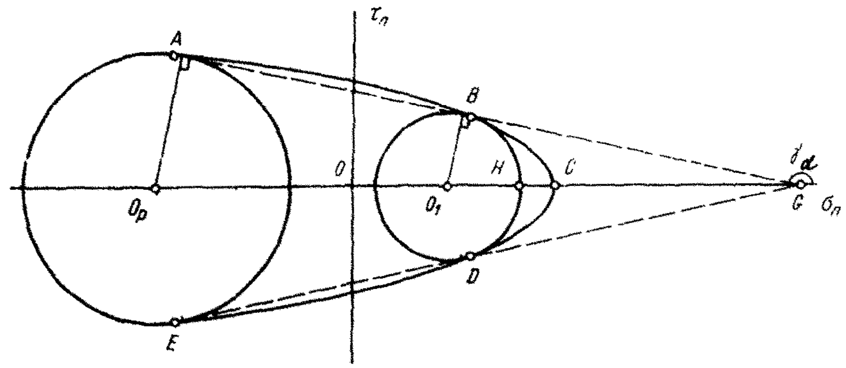

Fig. 3.

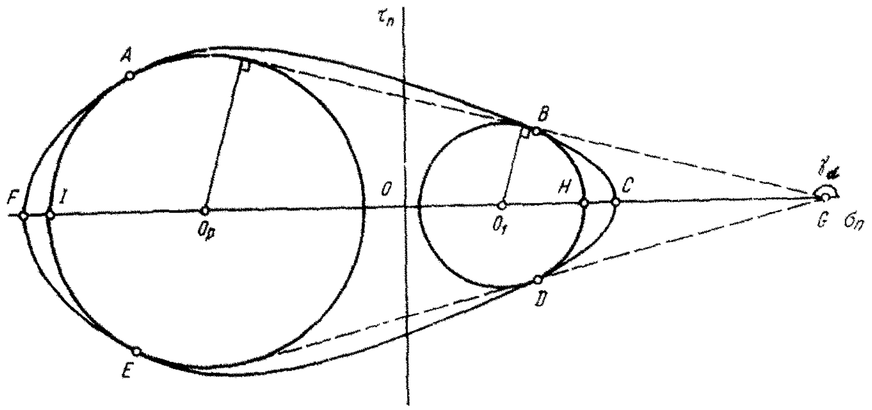

Fig. 4.

hummocking (Fig. 3) and before transition to a hummocky state (Fig. 4). The point O_p has coordinates $(-\pi_p, O)$.

It is not convenient to use the plasticity condition (1.7) with an arbitrary function F to solve (1.1)–(1.4), because the condition for the Mohr circle to touch the curve (1.7) can be expressed in terms of the variables p, s, τ appearing in (1.1)–(1.4) only through some rather lengthy algebra. At the same time, condition (1.7) is physically quite intuitive. We will therefore propose a simple method for approximating the curve (1.7), which intuitively fits the properties of ice covers and leads to an easier formulation of condition (1.7) in terms of p, s and τ.

On the arcs AB and ED in Figs 3 and 4, F is approximated by a linear function, corresponding to plastic shearing in a fine-grained friable medium [10]

$$|\tau_n| = (\sigma_n + \pi_d)\mathrm{tg}\gamma_d, \; \pi_p > p > \pi_1$$
$$\pi_d = \pi_d(A, h) < 0, \; \gamma_d = \gamma_d(A) \tag{1.10}$$

The point G in Figs 3 and 4 has coordinates $(-\pi_d, 0)$. Using (1.8) and (1.9), we can write condition (1.10) in the form

$$s^2 + \tau^2 = \sin^2\gamma_d(p - \pi_d)^2, \; \pi_p > p > \pi_1 \tag{1.11}$$

The arcs BCD in Figs 3 and 4 and AFE in Fig. 4 are approximated by the arcs BHD and AIE of the Mohr circles O_1 and O_p, respectively. The conditions for the Mohr circle to touch the circles O_1 and O_p, respectively, may be written in the form

$$\sigma_{max} = \pi_{cr} > 0,\ \ \pi_1 \geqslant p \geqslant -\pi_{cr} \tag{1.12}$$

$$\sigma_{min} = \pi_r < 0, -\pi_r \geqslant p \geqslant \pi_p \tag{1.13}$$

$$\pi_{cr} = -\pi_1 + (\pi_1 - \pi_d)\sin\gamma_d,\ \pi_r = -\pi_p - (\pi_p - \pi_d)\sin\gamma_d$$

Using (1.9), we can rewrite conditions (1.12) and (1.13) as follows:

$$s^2 + \tau^2 = (p + \pi_{cr})^2, \pi_1 \geqslant p \geqslant -\pi_{cr} \tag{1.14}$$

$$s^2 + \tau^2 = (p + \pi_r)^2, -\pi_r \geqslant p \geqslant \pi_1 \tag{1.15}$$

Thus, the conditions of plastic shearing, breakage and transition to a hummocky state are (1.11), (1.14) and (1.15), respectively. When conditions (1.14) and (1.15) are satisfied in an element of the ice cover, we must assume that

$$p = s = \tau = \pi_1 = \pi_d = 0 \tag{1.16}$$

$$p = \pi_p(A, h, h_f), h_f = h,\ s = \tau = \pi_1 = \pi_d = 0 \tag{1.17}$$

Elastic compression stretching deformations occur when

$$s^2 + \tau^2 < (p - \pi_{cr})^2, \pi_p \geqslant p > -\pi_{cr} \tag{1.18}$$

in ice covers of type 1 and when

$$s^2 + \tau^2 \ < \min[(p + \pi_{cr})^2, (p + \pi_r^2)]\ , -\pi_r > p > -\pi_{cr} \tag{1.19}$$

in ice covers of type 2. Inequality (1.18) can become an equality only if

$$d\rho/dt \leqslant 0 \tag{1.20}$$

If either of conditions (1.18) and (1.19) holds, we must assume that

$$p = \pi_e(\rho, A, h),\ dA/dt = 0,\ dh/dt = 0 \tag{1.21}$$

For a hummocky ice cover, the function π_e also depends on h_f. The functions $\rho_{cr}(A, h)$ in (1.5) and (1.6) are determined from the condition $\pi_e(\rho, A, h) = \pi_p(A, h)$.
An elastic shearing deformation occurs when

$$s^2 + \tau^2 \leqslant \sin^2(\gamma_d)(p - \pi_d)^2 \tag{1.22}$$

Equality in the first of these relationships can occur only if, once λ has been eliminated from (1.3) using (1.11), it turns out that

$$\lambda \leqslant 0 \tag{1.23}$$

When conditions (1.22) and (1.23) hold, we must put

$$\lambda = 0 \tag{1.24}$$

in (1.3).
The ice-cover model proposed above involves eight unknown functions ρ, A, h, s, τ, p, u, v, describing the macroscopic structure of the ice cover and depending on x, y and t. These

functions are given at the initial time $t=0$ in the (x, y) plane and are determined from (1.1)–(1.4) at subsequent times.

2. In ice-cover dynamics one typically has to determine the evolution of the macroscopic parameters $\rho, A, h, u, v, s, \tau, p$ in some region Ω of the (x, y) plane at times $t > 0$, given the stresses or displacements on the boundary of Ω and initial data at $t=0$. The variation of the macroscopic parameters may be determined from (1.1)–(1.4) provided that the functions

$$A_*, A_{**}, h_f, \pi_1, \pi_d, \pi_e, \pi_p, \gamma_d \tag{2.1}$$

characterizing the rheological properties of the ice cover are known at each point of Ω.

The functions (2.1) depend on parameters that define the macrostructure of the ice cover (the geometrical dimensions, shapes and physical–mechanical properties of the ice floes and their relative positions), which in turn depend on the temperature of the atmosphere and the ocean, the heat flux, solar radiation, and so on. Exact theoretical determination of these functions is, therefore, almost impossible. Nevertheless, by expressing them in a simple form, making natural assumptions about the properties of ice covers, one can qualitatively describe certain effects in sea ice dynamics.

Let us consider some simple approximations for the functions (2.1).

Suppose first that $1/2 < A_* \leqslant A_{**} \leqslant 1$. If

$$A_* = A_{**} = 1 \tag{2.2}$$

stresses will occur only in a consolidated pack ice cover; if $A < 1$, the cover will be in the diverging state with stress tensor identically equal to zero. The approximation (2.2) may be used if the functions π_d and γ_d have pronounced extrema at $A=1$; $\gamma_d \equiv \pi$, $\pi_p \equiv 0$ if $A < 1$; $\pi_1 = \pi_d = 0$. These assumptions are legitimate, e.g. in the case of a strongly crushed ice cover, made up of small ice floes.

For the functions π_e we assume that

$$\pi_e = k_e(A)h(\rho - \rho_0)/\rho_0 \tag{2.3}$$

for a non-hummocky ice cover and

$$\pi_e = k_e(A)\theta(\alpha)h(\rho - \rho_0)/\rho_0$$
$$\alpha = 1 - h_f/h,\ 0 \leqslant \theta(\alpha) \leqslant 1 \tag{2.4}$$

for a hummocky ice, where ρ_0 is the density of ice in the undeformed state. The function $\theta(\alpha)$ characterizes the compaction of the ice floes in a hummock and their degree of cohesion on freezing. If $\theta(\alpha)=1$ formulae (2.3) and (2.4) are identical, corresponding to the case in which freezing converts a hummock into a consolidated pack ice cover. The parameter α is a measure of roughness.

For the plastic deformations of compaction and hummocking, we set

$$\pi_p = k_p(A)h$$

in the case of a non-hummocky ice cover and

$$\pi_p = k_p^r(A)\theta(\alpha)h$$

in the case of a hummocky ice cover. A good approximation is $k_e(A) = E/(1-v^2)$, where E and v are Young's modulus and Poisson's ratio of the ice. In other words, we are assuming in this case that a close pack ice cover subjected to compression or stretching deformations behaves like a linearly elastic Hooke's body with the parameters of consolidated pack ice.

As the functions $k_p(A)$, $k_p^r(A)$, $\gamma_a(A)$, $\theta(\alpha)$ are defined in the interval (0., 1), they can be conveniently approximated by simple functions, whose parameters are selected by comparing numerical computations with experimental observation. For example, some researchers have used the expression [5]

$$k_p^r(A) = k_p(A) = P_* \exp[20(A-1)],\ P_* \cong 5\times10^3\ \text{N/m}^2$$

The functions π_1 and π_d may take arbitrary non-positive values, but π_σ should not exceed the tensile strength of ice [11]. At $\pi_1 = \pi_d = 0$ the ice cover no longer resists stretching but goes into the diverging state.

For a hummocky ice cover with $h \cong h_f$, as shown in Fig. 2, one can set

$$\gamma_d = \pi \tag{2.5}$$

Hence it follows that $s = \tau = 0$. In that case the ice cover is modelled by a continuous medium: a compressible fluid with possible irreversibility of compression–stretching deformations. This case agrees entirely with the model considered in [5].

In a close pack ice cover one assumes that

$$(\mu, k_e) \gg (s, \tau, p) \tag{2.6}$$

which follows from the assumption that the real internal stresses in the sheet are much less than its elastic moduli [12].

3. It is known that wind, sea currents and waves interact with a sufficiently open pack ice cover to induce the formation of various structures of more closely packed ice, such as cells, eddies, strips, etc. [6, 13, 14]. To describe possible mechanisms for these processes, let us consider a diverging ice cover drifting in the field of an ocean eddy.

Express the force **F** in (1.2) as

$$\mathbf{F} = \mathbf{F}_{ai} + \mathbf{F}_{wi} - \rho g h\, \nabla\eta \tag{3.1}$$

where $\mathbf{F}_{ai}$ and $\mathbf{F}_{wi}$ are the forces of friction of the wind and the current, respectively, on ice [6]

$$\mathbf{F}_{ai} = C_a\rho_a|\mathbf{V}_a|\mathbf{V}_a,\ \mathbf{F}_{wi} = C_w\rho_w(\mathbf{u}_w - \mathbf{u}) \tag{3.2}$$

$$\mathbf{V}_a = (V_{a,x}, V_{a,y}),\ \mathbf{u}_w = (u_{w,x}, u_{w,y})$$

$C_{a,w}$ are the coefficients of friction, ρ_a and ρ_w are the densities of air and water, and $\mathbf{V}_w$ and $\mathbf{u}_w$ are the wind and current velocities, respectively. The equation $z = \eta(x, y, t)$ describes the difference between the ocean surface and its horizontal equilibrium position.

Large time-scale quasi-steady motions of a diverging ice cover are described by the system of equations

$$dh/dt = 0,\ dA/dt + A\nabla\mathbf{u} = 0,\ \rho h\mathbf{f}\times\mathbf{u} = \mathbf{F} \tag{3.3}$$

Steady motions of the ocean corresponding to time scales in the f-plane approximation are described by the shallow-water equations [15]

$$\mathbf{f}\times\mathbf{u}_w = -g\nabla\eta,\ \nabla\mathbf{u}_w = 0$$

Here we are ignoring the effect of the wind and the ice cover on the velocity field in the liquid compared with flows present in the liquid when there is no wind and pack ice. This is equivalent to viewing a diverging ice cover as a passive tracer of hydrothermodynamic processes in the ocean and the atmosphere [16].

Introducing the stream function Ψ_w for the velocity field in the liquid, we obtain

$$u_w = -\partial\Psi_w/\partial y,\ v_w = \partial\Psi_w/\partial x,\ \eta = f\Psi_w/g$$

In turbulent flow one has a stream function $\Psi_w = \Psi_w(r)$, $r = (x^2+y^2)^{1/2}$ of the shape shown in Fig. 5. It follows from (3.2) and (3.3) that the drift velocity field of the ice cover has a stream function

$$\Psi = \Psi_w + \alpha x - \beta y$$
$$\alpha = \Delta(V_{a,x}\rho hf - V_{a,y}C_w\rho_w),\ \beta = \Delta(V_{a,x}C_w\rho_w + V_{a,y}\rho hf)$$
$$\Delta = C_a\rho_d |V_d| [(C_w\rho_w)^2 + (\rho hf)^2]^{-1}$$

The singular points of Ψ in the (x, y) plane are determined from the relations

$$y = \beta x/\alpha,\ |d\Psi_w/dr| = (\alpha^2+\beta^2)^{1/2}$$

If Ψ_w has the shape shown in Fig. 5 and moreover

$$(\alpha^2+\beta^2)^{1/2} < \max_r |d\psi_w/dr| \quad (3.4)$$

then there are two singular points in the (x, y)-plane. If (3.4) becomes an equality, the two singular points coalesce; if the inequality sign is reversed, there are no singular points.

The equations of the streamlines, that is, the trajectories of the ice-cover elements, have the form

$$dx/dt = -\partial\Psi/\partial y,\ dy/dt = \partial\Psi/dx \quad (3.5)$$

To determine the type of singular point, let us place the origin at the point and represent Ψ in its neighbourhood in the form

$$\Psi = \Psi^i_{xx}x^2 + \Psi^i_{xy}xy + \Psi^i_{yy}y^2,\ i = 1,2 \quad (3.6)$$

where Ψ^i_{xx}, Ψ^i_{xy}, Ψ^i_{yy} are the values of the appropriate derivatives of Ψ at the ith singular point. Substituting (3.6) into (3.5) and using (3.4), we find the eigenvalues of system (3.5)

$$\lambda_i^2 = (\Psi^i_{xy})^2 + \Psi^i_{xx}\Psi^i_{yy} = -(d\Psi_w/dr)^i (d^2\Psi_w/dr^2)^i r^{-1}$$

Hence, consulting Figs 5 and 6, we see that the singular point O_1 is a centre and O_2 is a saddle point. The phase portrait is shown in Fig. 6.

The distribution of compactness in this case is described by the transport equation $dA/dt = 0$. If there was initially no ice cover within the region Ω in the (x, y) plane bounded by the separatrix passing through O_2, then no ice cover will appear there as time passes. And vice versa: an ice cover inside Ω cannot leave that region. Its motion within Ω is turbulent in nature. An ice cover below the phase curve through O_2 will describe strongly curved trajectories in the neighbourhood of the separatrix loop.

4. Let us consider a homogeneous ice cover floating on the surface of a viscous liquid under the action of a constant wind, assuming that the surface of the liquid is horizontal. It will be convenient to denote

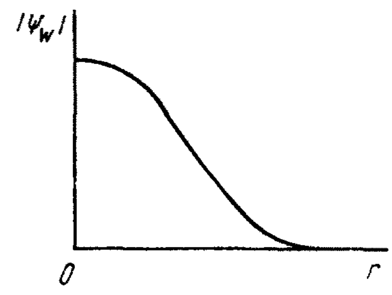

Fig. 5.

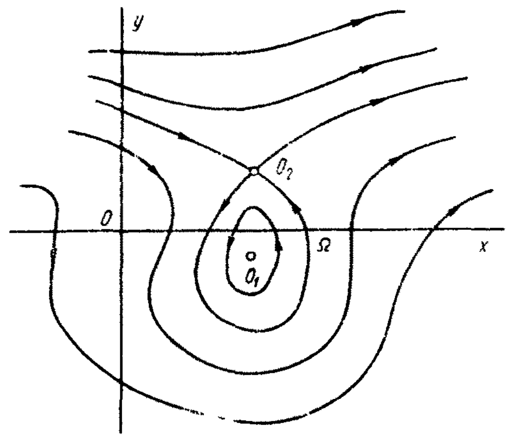

Fig. 6.

$$u_w = u_{w,x} + iu_{w,y},\ V_a = V_{a,x} + iV_{a,y},\ V = u + iv$$

$$F_{ai} = F_{ai,x} + iF_{ai,y},\ F_{wi} = F_{wi,x} + iF_{wi,y},\ i^2 = -1$$

The equations of steady motion for the liquid and the ice cover may be written in complex form as follows (μ denotes the viscosity of the liquid)

$$\mu\partial^2 u_w/\partial z^2 = i\rho_w f u_w,\ i\rho h f V = F_{ai} + F_{wi} \tag{4.1}$$

The boundary conditions on the liquid surface $z=0$ and at the bottom $z=-H$ will be

$$\mu(Au_w/h_E + (1-A)\partial u_w/\partial z) = \mu AV/h_E + (1-A)F_{aw},\ z = 0 \tag{4.2}$$

$$u_w = 0,\ z=-H;\ h_E = (f\rho_w/\mu)^{-1/2}$$

At $A=1$ (consolidated pack ice), condition (4.2) reduces to the requirement that the liquid particles adhere to the ice. In the absence of an ice cover, $A=0$, condition (4.2) simply specifies the wind-shear stresses on the surface of the liquid

$$F_{aw} = C_{aw}\rho_a|V_a|V_a$$

If the depth H of the liquid substantially exceeds the Ekman depth h_E, which is typically a few dozen metres [15], the solution of (4.1) will have the following form near the liquid surface [17]

$$u_w = k\exp(\lambda z)/\lambda,\ \lambda = \exp(i\pi/4)/h_E$$

The complex numbers V and k are found from the second equation of (4.1) and (4.2)

$$V = \Delta[C_a h_E(A\exp(-i\pi/4) + 1 - A) + C_{aw}(1-A)h_E C_w\rho_w/(\mu\lambda)] \tag{4.3}$$

$$k = \Delta[C_a A + (1-A)C_{aw}h_E(i\rho hf + C_w\rho_w)/\mu]$$

$$\Delta = \rho_a|V_a|V_a[AC_w\rho_w/\lambda + (A\lambda + h_E(1-A))(i\rho hf + C_w\rho_w)]$$

In the special cases considered above, we obtain from (4.4)

$$\nabla\lambda = k = C_a\rho_a|V_a|V_a(2C_w\rho_w + i\rho hf),\ A = 1 \tag{4.4}$$

$$V = \rho_a|V_a|V_a[C_a + C_{aw}C_w\rho_w/(\mu\lambda)][C_w\rho_w + i\rho hf]^{-1}$$

$$k = C_{aw}\rho_a|V_a|V_a/\lambda, A = 0$$

In the limiting case $A \to 0$ the velocity of the liquid at the surface $z = 0$ makes an angle $\pi/4$ with the wind direction [15]. The velocity of the ice cover is determined by the balance of the frictional forces due to the wind and the liquid and the Coriolis force. In a drifting consolidated pack ice cover ($A = 1$) the angle between the wind velocity and the ice cover velocity is $\pi/4 + \mathrm{arctg}[\rho hf/(2C_w\rho_w)]$.

It follows from (4.3) and (4.4) that sections of the ice cover with differing thicknesses and compactnesses will drift at different velocities. In the same wind, therefore, they may collide, overtaking one another and forming regions of close pack ice where the collisions occur, or drift apart leaving regions of open water. If the boundaries of these sections are straight lines, strips of close pack ice will form when they collide. The formation and evolution of strips may be described by discontinuous solutions, of the "filament" type, of the equations of dynamics for a diverging ice cover [18].

5. We will now investigate the effect of plane surface waves on the distribution of a diverging ice cover over the ocean surface. The motions under consideration are typified by time scales much shorter than 24 hours. Hence the Coriolis force may be ignored in the momentum equations. The laws of conservation of mass and momentum may be written characteristically as

$$dA/dt + A\partial u/\partial x = 0, \quad dx/dt = u \tag{5.1}$$

$$\rho h du/dt = C_w\rho_w(u_w - u) - \rho g h \partial\eta/\partial x + C_a\rho_a|V_a|V_a$$

It is clear from (5.1) that the elements of the ice cover move along the characteristics. Intersections of characteristics correspond to regions of ice accretion.

Let us assume that the flow of the liquid has a velocity potential φ. Then, for linear waves

$$u = \partial\varphi/\partial x, \quad \partial\eta/\partial t = \partial\varphi/\partial z, \quad z = 0 \tag{5.2}$$

As an example, let us consider the potential of periodic plane waves in an infinitely deep liquid

$$\varphi = u_0 \sin\theta e^{kz}, \quad \omega^2 = gk, \quad \theta = kx + \omega t, \quad k > 0 \tag{5.3}$$

where u_0 is the amplitude of the horizontal velocity of the liquid particles in the wave. The wave amplitude is $a = u_0\omega/g$.

Substituting (5.2) and (5.3) into (5.1), we obtain

$$\rho h du/dt = \alpha\sin(\theta_0 - \theta) - C_w\rho_w u + C_a\rho_a|V_a|V_a \tag{5.4}$$

$$d\theta/dt = \omega(\omega u/g + 1)$$

$$\alpha = u_0[(C_w\rho_w)^2 + (\rho\omega h)^2]^{1/2}, \quad \sin\theta_0 = C_w\rho_w u_0/\alpha$$

System (5.4) may have singular points $(-g/\omega, \theta_{1,2}^n)$ in the (u, θ) plane, where

$$\theta_1^n = \theta_0 + \arcsin(\beta/\alpha) + 2n\pi, \quad n \in Z \tag{5.5}$$

$$\theta_2^n = \theta_0 - \arcsin(\beta/\alpha) + (2n + 1)\pi$$

$$\beta = C_w\rho_w g/\omega + C_a\rho_a|V_a|V_a$$

The condition $\beta/\alpha \leq 1$ for the existence of singular points may be written as an inequality for the wave amplitude

$$a \geqslant (C_w\rho_w g + C_a\rho_a\omega|V_a|V_a)[g^2((C_w\rho_w)^2 + (\rho\omega h)^2)]^{-1/2} \tag{5.6}$$

Linearizing Eqs (5.4) in the neighbourhood of the singular points (5.5) and evaluating the eigenvalues corresponding to θ_1^n, θ_2^n, we obtain

$$2\lambda_{1,2} = -C_w\rho_w \pm D_{1,2}^{1/2} \tag{5.7}$$
$$D_{1,2} = (C_w\rho_w)^2 + (-1)^{1,2}\omega^2(\alpha^2 - \beta^2)/g$$

The singular points θ_2^n are saddle points θ_1^n are stable foci if $D_1 < 0$ and nodes if $D_1 \geqslant 0$. The existence of foci or nodes in the phase plane (u, θ) implies that there are regions in the (θ, t) plane in which characteristics intersect. In that case the waves induce the formation of strips in the ice.

If the wind and wave velocities are in the same direction and the wave amplitude a is sufficiently small, then $\omega V_a < 0$, and condition (5.6) may hold only when the wind is strong enough. In that case intensification of the wind will cause more intensive formation of ice strips.

6. We will now consider the effect of a current in the liquid, localized in the y direction, on a close pack ice cover. The current velocity is in the direction of the x axis. The current velocity profile $V(y)$ is shown in Fig. 7. The frictional force exerted by the liquid on an element of the ice cover in the x direction is determined by the formula

$$F = C_w(V(y) - u) \tag{6.1}$$

Let us assume that the ice cover is stationary and that the current generates elastic stresses in the cover. Then (1.1), (1.2) reduce to the equations of statics

$$-\frac{\partial p}{\partial x} + \frac{\partial s}{\partial x} + \frac{\partial \tau}{\partial y} = -F, \quad -\frac{\partial p}{\partial x} - \frac{\partial s}{\partial y} + \frac{\partial \tau}{\partial x} = 0, \quad A = \text{const}, \quad h = \text{const} \tag{6.2}$$

to which we must add compatibility conditions for the deformations. Taking Hooke's law (2.5) and (2.9) for compressive deformations, we obtain

$$p = -k(\partial w_x/\partial x + \partial w_y/\partial y), \; k = k_e h$$

where $w_{x,y}$ are the components of the displacement in the x and y directions, respectively. The condition for the deformations to be consistent may be written as [19]

$$2\frac{\partial^2\tau}{\partial x\partial y} + \frac{\mu + k}{2k}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)p - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)s = 0$$
$$\left(\tau = \mu\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right), \; s = \mu\left(\frac{\partial w_x}{\partial x} - \frac{\partial w_y}{\partial y}\right)\right) \tag{6.3}$$

Eqs (6.2) and (6.3) have a simple solution

$$p = \sigma y + \sigma_p, \; s = -\sigma y + \sigma_s, \; \frac{\partial \tau}{\partial y} = -F, \; \tau = \mu\frac{\partial w_x}{\partial y} \tag{6.4}$$
$$\sigma = \text{const}, \; \sigma_p = \text{const}, \; \sigma_s = \text{const}$$

Let us consider the effect of an under-ice current on the edge of the ice cover, which is represented by a straight line $y=y_1$. The liquid surface, where $y<y_1$, is free of ice. The boundary condition for the stresses is $\tau=0$ at $y=y_1$. Hence, by (6.4), we have

$$\tau=-\int_{y_1}^{y} F dy \tag{6.5}$$

If $y_1<H$ (see Fig. 7), then $\tau=\tau(y)$ is a monotone function (see Fig. 8) and $\tau=\text{const}$ for $y>H$.

Consider the interaction of a symmetrical under-ice current ($y_{max}=0$ in Fig. 7) with the ice cover, on the assumption that $w_x=0$, $y=\pm H$. The solution is given by the formulae

$$\tau=-\int_0^y F dy, \quad w_x=w_0+\int_0^y \tau dy \mu^{-1} \tag{6.6}$$

The constant w_0 is determined from the boundary conditions. Typical plots of $\tau(y)$ and $w_x(y)$ are shown in Fig. 9.

It follows from (6.5) and (6.6) that in the cases considered above, the maximum stresses are reached at the points $y=\pm H$. If the currents are sufficiently strong, therefore, the ice cover may experience plastic flow.

Let us consider steady plastic flows in which all the parameters depend only on y and $\nu=0$. By (1.1)–(1.3)

$$\frac{\partial \tau}{\partial y}=-F, \quad \frac{\partial (p+s)}{\partial y}=0 \tag{6.7}$$

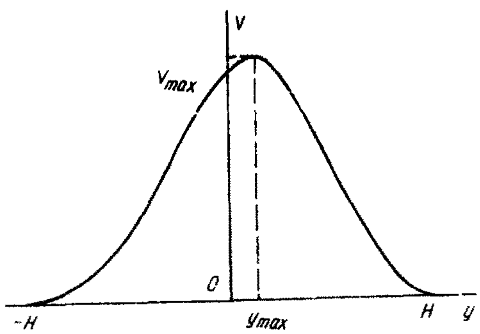

Fig. 7.

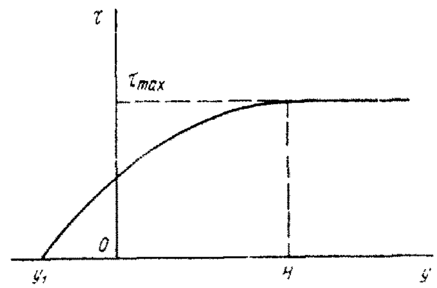

Fig. 8

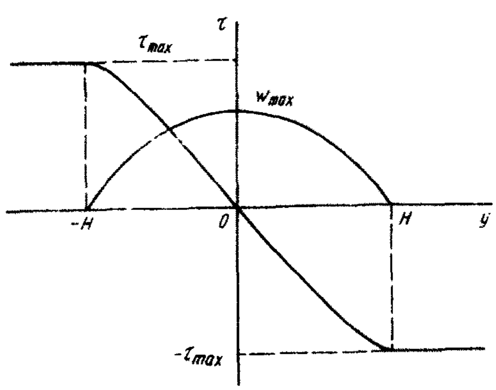

Fig. 9.

$$-\tau\frac{\partial u}{\partial y}+\lambda s=0,\quad s\frac{\partial u}{\partial y}+\lambda\tau=\mu\frac{\partial u}{\partial y} \tag{6.8}$$

$$s^2+\tau^2=\sin^2\gamma_d(p-\pi_d)^2 \tag{6.9}$$

$$\lambda\geq 0 \tag{6.10}$$

Assuming that $\partial u/\partial y\neq 0$, we deduce from (6.8) that

$$s^2+\tau^2=\mu s,\ p=-s-s_0 \tag{6.11}$$

It follows from the first equation of (6.8), (6.9) and (6.11) that

$$p=\text{const},\ s=\text{const},\ \tau=\text{const},\ u=V(y) \tag{6.12}$$

The plasticity condition (6.9) reduces to the equation

$$\sin^2\gamma_d(s+s_0+\pi_d)^2-\mu(s+s_0+\pi_d)+\mu(s_0+\pi_d)=0 \tag{6.13}$$

The solution of Eq. (6.13) satisfying the estimates (2.6) may be written approximately as

$$s+s_0+\pi_d\equiv s_0+\pi_d+\sin\gamma_d\frac{(s_0+\pi_d)^2}{4\mu}+O(\frac{(s_0+\pi_d)^2}{\mu^2}) \tag{6.14}$$

Hence it is obvious that $s\geq 0$ and condition (6.10) will hold if $\tau\partial u/\partial y\geq 0$. The constant s_0 in the solution determines the pressure p. The stress τ is found from (6.9).

Let us assume that the velocity field $V(y)$ and the stresses s and τ are such that problems (6.5) and (6.6) cannot possibly have a purely elastic solution. We will construct steady elastic–plastic solutions, on the assumption that part of the ice cover is moving as a whole at a constant speed $u_0=\text{const}$, while the other part is taking part in plastic flow. The stresses in the elastic region must reach a limiting maximum value $\tau=\pm\tau_p$ that satisfies the plasticity condition (6.9) at the boundaries with the plastic region.

In the problem of the interaction of a current with the ice edge, the elastic region has a left boundary coinciding with the edge. Its width L and u_0 are found from the equations

$$\tau_p=C_w\int_{y_1}^{L}(V(y)-u_0)dy,\quad V(L)=u_0$$

The plastic flow (6.12) and (6.13) borders on the right edge of the elastic region. The stresses in the elastic region are determined from (6.6), where we must assume that $u=u_0$. The velocity profile of the ice cover is shown in Fig. 10.

In the interaction of a symmetrical current with a consolidated pack ice cover, the elastic region is a strip with boundaries $y=\pm L$. The quantities u_0 and L are found from the equations

$$\tau_p=C_w\int_{0}^{L}(V(y)-u_0)dy,\quad V(L)=u_0$$

The region of the plastic flow (6.12) and (6.13) borders on the elastic region. The stresses in the elastic region are determined from (6.6), where we must assume that $u=u_0$. The velocity profile of the ice cover is shown in Fig. 11. The width of the elastic region is $2L$ and the velocity u_0 depend on τ_p, hence also on the stresses p and s. Hence an increase in ice compression will induce an increase in τ_p and may suppress the plastic flow.

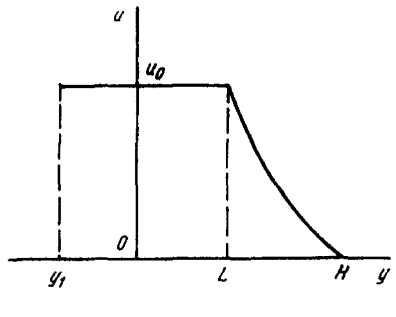

Fig. 10.

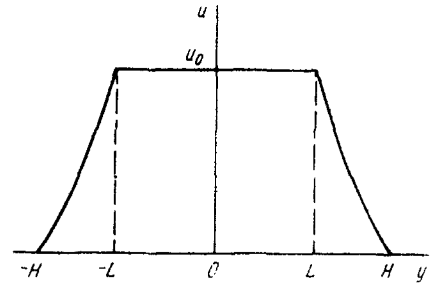

Fig. 11.

7. We will now consider a hummocky ice cover with $h \equiv h_f$. Let us assume that the approximations (2.2) and (2.5) hold, i.e. the ice cover is compacted up to $A = 1$ without the formation of internal stresses, and that the shearing stresses in hummocking are much less than the compression. The system of equations for the motion of such an ice cover is

$$\partial(Ah)/\partial t + \nabla(Ah\mathbf{u}) = 0 \tag{7.1}$$

$$\rho_0 Ah\, d\mathbf{u}/dt = -\nabla p + \mathbf{F} \tag{7.2}$$

$$p = 0 \quad \text{if} \quad A < 1;\ p = \pi_p(h, h_f) \quad \text{if} \quad dh/dt > 0, A = 1 \tag{7.3}$$

$$p = \pi_e,\ dh/dt = 0 \quad \text{if} \quad \pi_l(h, h_f) < p < \pi_p(h, h_f), A = 1$$

$$\pi_p(h_f, h_f) = \pi_l(h_f, h_f) = 0,\ \pi_p(h, h_f) = k(h - h_f)$$

Consider the problem of an ice massif colliding with a solid wedge-shaped wall with wedge angle 2α. The sides of the wedge make angles α with the x axis.

This problem was considered for $\alpha = \pi/2$ in [1], where an unsteady, discontinuous solution describing the process was found. We shall investigate steady discontinuous solutions of equations (7.1)–(7.3) with $\mathbf{F} = 0$, which may occur in reality in flow past a wedge with angle $2\alpha < 2\alpha_* < \pi$ near the apex, where the influence of the force $\mathbf{F}$ defined by (3.1) is small compared with the internal stresses in the ice.

Let us assume that the ice cover, on collision with the wall, is compacted up to $A = 1$ and then forms hummocks. At the stationary fracture between the diverging and close packed hummocky ice we have

$$A_1 h_1 V_1 \sin\theta = h_2 V_2 \sin(\theta - \alpha),\quad V_1 \cos\theta = V_2 \cos(\theta - \alpha) \tag{7.4}$$

$$A_1 h_1 V_1^2 (1 - A_1 h_1 / h_2) \sin^2\theta = \pi_p(h_2, h_1)/\rho_0$$

The subscript 1 indicates the parameters of the diverging ice cover entering the fracture, the subscript 2 indicates the parameters of the hummocky ice cover and θ is the angle between the direction of the discontinuity and the x axis.

From (7.4) we derive a cubic equation for θ

$$(z - \upsilon)[z(A_1\mu + 1) + 1] = A_1 z (1 + z)^2 \tag{7.5}$$

$$z = \mathrm{tg}\theta\,\mathrm{tg}\alpha,\quad \upsilon = \mathrm{tg}^2\alpha,\quad \mu = \rho_0 V_1^2 / k$$

Equation (7.5) always has one real root $z^- < 0$, which is physically meaningless. It follows from (7.5) that if there are two more real roots $z_{1,2}^+$, they will satisfy the condition $z_{1,2}^+ > \mathrm{tg}^2\alpha$.

The existence of positive roots $z_{1,2}^+$ depends on the sign of the determinant of Eq. (7.5), which is too cumbersome to present here. Vanishing of the determinant corresponds to the limiting angle of the wedge $2\alpha_*$ at which there is a solution with a stationary discontinuity attached to the wedge tip. At $\alpha < \alpha_*$ there are two solutions with an attached discontinuity, both such that $\theta > \alpha$. An analogous effect occurs in gas dynamics in flow past a wedge [20]. The unique solution may be determined from experimental data.

Let us consider two limiting cases in which a solution of Eq. (7.5) may be constructed approximately. Let $\mu \ll 1$, that is, the ice cover is moving slowly. Then

$$z^- = -1 + O(\mu),\quad z_{1,2}^+ = (1 - A_1 \pm \sqrt{\Delta_1})/(2A_1) + O(\mu) \tag{7.6}$$

$$\Delta_1 = 1 + A_1^2 - 2A_1(1 + 2\mathrm{tg}^2\alpha)$$

The condition $\Delta_1 \geqslant 0$ yields the inequality

$$A_1 \leqslant N(\alpha),\quad N(\alpha) = 1 + 2\mathrm{tg}^2\alpha - 2\mathrm{tg}\alpha(1 + \mathrm{tg}^2\alpha)^{1/2}$$

In other words, for a solution with attached discontinuity to be feasible at slow ice speeds, the compactness must be less than $N(\alpha) \leqslant 1$. The maximum angle of the wedge is determined in that case by the formula

$$\mathrm{tg}^2\alpha_* = \frac{1}{2}\left(\frac{1 + A_1^2}{2A_1} - 1\right)$$

Now set $\upsilon \ll 1$, corresponding to a small wedge angle. Then

$$z_1^+ = \upsilon/(1 - A_1) + O(\upsilon^2),\quad z^- = (1 + \mu A_1 - 2A_1 - \sqrt{\Delta_2})/(2A_1) + O(\upsilon) \tag{7.7}$$

$$z_2^+ = (1 + \mu A_1 - 2A_1 + \sqrt{\Delta_2})/(2A_1) + O(\upsilon),\quad \Delta_2 = (1 + \mu A_1 - 2A_1)^2 + 4A_1(1 - A_1)$$

Obviously, if A_1 is close to unity, the solution (7.7) becomes meaningless. So let $A_1 = 1 - \sqrt{(\upsilon)}a$. Then the root z_1^+ may be found as a series in powers of $\sqrt{(\upsilon)}$

$$z_1^+ = \sqrt{\upsilon}y_1 + \upsilon y_2 + \ldots,\quad y_1 > 0 \tag{7.8}$$

Substituting (7.8) into (7.5), we obtain

$$(\mu - 1)y_1^2 + ay_1 - 1 = 0 \tag{7.9}$$

This equation has real roots provided that $a^2 > 4(1 - \mu)$. If $\mu \geqslant 1$ there is a real and positive root y_1. If $\mu < 1$ the condition for Eq. (7.9) to have real roots becomes

$$A_1 \leqslant N(\alpha, \mu),\quad N(\alpha, \mu) = 1 - 2\mathrm{tg}\alpha\sqrt{1 - \mu}$$

Hence it is clear that the critical compactness of the ice, at which solutions may exist with an attached discontinuity, increases together with the velocity at which the ice approaches the wedge.

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